R300 – Advanced Econometric Methods

PROBLEM SET 4 - SOLUTIONS

Posted on Mon. November 4 Due on Tue. November 12, 2018 at noon

1. Suppose that

$$y_i = x_i\beta_1 + z_i\beta_2 + \varepsilon_i.$$

State and show the Frisch-Waugh-Lovell theorem (say, for β_1) by explicit calculation.

The Frisch-Waugh-Lovell theorem here goes as follows. Regressing y_i on z_i gives residuals

$$\hat{u}_i = y_i - z_i\hat{\gamma}_1, \qquad \hat{\gamma}_1 = \frac{\sum_i z_i y_i}{\sum_i z_i^2}.$$

Regressing x_i on z_i gives residuals

$$\hat{v}_i = x_i - z_i\hat{\gamma}_2, \qquad \hat{\gamma}_2 = \frac{\sum_i z_i x_i}{\sum_i z_i^2}.$$

Regressing $\hat{u}_i$ on $\hat{v}_i$ gives slope coefficient

$$\hat{\beta}_1 = \frac{\sum_i \hat{v}_i\hat{u}_i}{\sum_i \hat{v}_i^2}.$$

This slope is numerically identical to the least-squares estimator applied directly to the bivariate regression problem.

To see this first note that the least-squares estimator solves

$$\min_{b_1,b_2} \sum_{i=1}^{n} (y_i - x_i b_1 - z_i b_2)^2.$$

The first-order conditions are

$$\sum_{i=1}^{n} x_i\,(y_i - x_i b_1 - z_i b_2) = 0,$$
$$\sum_{i=1}^{n} z_i\,(y_i - x_i b_1 - z_i b_2) = 0.$$

From the second equation,

$$\sum_i z_i(y_i - x_i b_1) = \sum_i z_i^2\, b_2$$

and so

$$b_2 = \frac{\sum_i z_i(y_i - x_i b_1)}{\sum_i z_i^2}.$$

Substituting back into the first-order condition for b_1 gives

$$\sum_{i=1}^{n} x_i \left(y_i - x_i b_1 - z_i \frac{\sum_{i'} z_{i'} y_{i'}}{\sum_{i'} z_{i'}^2} + z_i \frac{\sum_{i'} z_{i'} x_{i'}}{\sum_{i'} z_{i'}^2}\, b_1 \right) = 0.$$

But note that (using the definitions from above) this is

$$\sum_{i=1}^{n} x_i\ \left((y_i - z_i \hat{\gamma}_1) - (x_i - z_i \hat{\gamma}_2)\, b_1 \right) = 0,$$

or even

$$\sum_{i=1}^{n} x_i\ (\hat{u}_i - \hat{v}_i\, b_1) = 0.$$

The solution is

$$\hat{\beta}_1 = \frac{\sum_i x_i\, \hat{u}_i}{\sum_i x_i \hat{v}_i} = \frac{\sum_i \hat{v}_i\, \hat{u}_i}{\sum_i \hat{v}_i^2}.$$

To see the last transition note that, by definition, $x_i = z_i \hat{\gamma}_2 + \hat{v}_i$. Then,

$$\sum_i x_i\, \hat{u}_i = \sum_i (z_i \hat{\gamma}_2 + \hat{v}_i)\, \hat{u}_i = \hat{\gamma}_2 \sum_i z_i\, \hat{u}_i + \sum_i \hat{v}_i\, \hat{u}_i = \sum_i \hat{v}_i\, \hat{u}_i.$$

Indeed, $\sum_i z_i\, \hat{u}_i = 0$ by properties of the least-squares regression of y_i on z_i. The same argument explains why $\sum_i x_i \hat{v}_i = \sum_i \hat{v}_i^2$.

Note that the regression of y_i on z_i is performed mostly for didactical purposes. Replacing $\hat{u}_i$ by y_i and estimating β_1 by

$$\frac{\sum_i y_i \hat{v}_i}{\sum_i \hat{v}_i^2}$$

gives the same result. (Be sure to convince yourself of this.)

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2. Suppose x is continuous and uniformly distributed on the interval $[\theta, \theta + 1]$. We wish to test

$$H_0 : \theta = 0 \quad \text{vs} \quad H_1 : \theta > 0.$$

Consider the procedure

$$\text{Reject } H_0 \text{ if } x > .95, \qquad \text{Accept } H_0 \text{ otherwise.}$$

(i) Compute the size of this test.

(ii) Derive the power function.

(i) Under the null, x is uniformly distributed on $[0, 1]$. Hence,

$$P_0(X > .95) = .05,$$

so that this test has size $\alpha = .05$.

(ii) Under the fixed alternative θ, x is uniformly distributed on $[\theta, \theta + 1]$ and so

$$P_\theta(x \leq v) = v - \theta.$$

The power function at θ is

$$\beta(\theta) := P_\theta(x > .95) = 1 - P_\theta(x \leq .95) = \begin{cases} 0 & \text{if } \theta \leq -.05 \\ \theta + .05 & \text{if } \theta \in (-.05, .95]) \\ 1 & \text{if } \theta > .95 \end{cases},$$

which is monotone non-decreasing in θ.

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2. Suppose $x \sim N(\mu, \sigma^2)$. Consider two independent random samples on x, $\{x_{1i}\}_{i=1}^n$ and $\{x_{2i}\}_{i=1}^n$. Find a sample size n so that

$$P\left(|\overline{x}_1 - \overline{x}_2| < \frac{\sigma}{5}\right)$$

is .99. Explain how you proceed.

By normality both sample means $\overline{x}_1$ and $\overline{x}_2$ have distribution $N(\mu, \sigma^2/n)$. By independence, their difference $\overline{x}_1 - \overline{x}_2$ is distributed as $N(0, 2\sigma^2/n)$. Now,

$$P\Big(|\overline{x}_1 - \overline{x}_2| < \frac{\sigma}{5}\Big) = P\Big(-\frac{\sigma}{5} < \overline{x}_1 - \overline{x}_2 < \frac{\sigma}{5}\Big) = P\Big(-\frac{1/5}{\sqrt{2}/\sqrt{n}} < z < \frac{1/5}{\sqrt{2}/\sqrt{n}}\Big)$$

for $z = (\overline{x}_1 - \overline{x}_2)/(\sqrt{2}\sigma/\sqrt{n})$ standard normal. Thus, we need to solve

$$P\Big(z \geq \frac{1}{5}\sqrt{\frac{n}{2}}\Big) = 1 - \Phi\Big(\frac{1}{5}\sqrt{\frac{n}{2}}\Big) = .005$$

for n. The table for critical values of the standard-normal distribution gives us that this is equivalent to solving $\frac{1}{5}\sqrt{\frac{n}{2}} = 2.576$ for n. This yields $n \approx 332$.

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4. Suppose that x_i is exponential with density

$$f_\theta(x) = \theta e^{-x\theta},$$

where $\theta > 0$ and $x \geq 0$.

(i) Derive the maximum likelihood estimator of θ, say $\hat{\theta}$.

(ii) Derive the asymptotic distribution of $\hat{\theta}$.

(iii) Derive the asymptotic distribution of the estimator $1/\hat{\theta}$.

(i) The MLE is $1/\overline{x}$.

(ii) The information bound is θ^2/n. The asymptotic distribution result is

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \theta^2).$$

(iii) Continuous mapping implies that $\overline{x} = 1/\hat{\theta} \xrightarrow{p} = 1/\theta = E(x_i)$. The Jacobian of the transformation is $-1/\theta^2$ and so, by the Delta method,

$$\sqrt{n}(\overline{x} - E(x_i)) \xrightarrow{d} N(0, 1/\theta^2)$$
